Hand Out 2

Routh Hurwitz Test & Jury's Test.

Continuous Time :—

$\dot{x} = Ax + Bu$

$y = Cx \,, \; x(0) = 0$

A is $n \times n$, B is $n \times 1$, C is $1 \times n$

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) \, d\tau .$$

Q: If $u(t)$, $t \geq 0$ is bounded, when is it true that $y(t)$, $t \geq 0$ is also bounded?

If all the eigenvalues of A show up in $y(t)$, then a bounded $u(t)$, $t \geq 0$ produces a bounded $y(t)$, $t \geq 0$ if and only if

All Eigenvalues of A have negative real part.

Discrete Time :-

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$$x_{k+1} = A x_k + B u_k$$

$$y_k = C x_k\ ,\ x_0 = 0$$

A is $n \times n$, B is $n \times 1$, C is $1 \times n$

$$y_k = \sum_{j=0}^{k-1} C A^j B u_{k-(j+1)}$$

Q: If $u(j), j \geq 0$ is bounded when is it true that $y(j), j \geq 0$ is also bounded?

If all the eigenvalues of A show up in y_k, then a bounded $u_j, j \geq 0$ produces a bounded $y_j, j \geq 0$ if and only if

All Eigenvalues of A have magnitude < 1.

Q: How do we ascertain if the eigenvalues of a matrix have

(a) Negative Real Parts.

(b) magnitude < 1.

The Routh Hurwitz stability criterion:

$$P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

Consider $P_n(\lambda)$ a monic polynomial with real co-efficients.

Form the determinants

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ 1 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & 1 & a_2 & \cdots & a_{2n-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

where $a_k = 0 \quad k > n$.

Theorem (Routh - Hurwitz)

Every zero of $P_n(\lambda)$ has a negative real part if and only if

$$\Delta_r > 0 \text{ for } r = 1, 2, \ldots, n.$$

Ex 1 $n=2$

$$P_2(\lambda) = \lambda^2 + a_1\lambda + a_2$$

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & 0 \\ 1 & a_2 \end{vmatrix}$$

Every zero of $P_2(\lambda)$ has a negative real part iff

$$\Delta_1 > 0 \ \& \ \Delta_2 > 0$$

$$\Rightarrow a_1 > 0 \ \& \ a_1 a_2 > 0$$

$$\Rightarrow a_1 > 0, \ a_2 > 0$$

Ex 2 $(n=3)$

$$P_3(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = a_1 a_2 - a_3$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3 \Delta_2$$

Every zero of $P_3(\lambda)$ has a negative real part iff

$$\Delta_1 > 0, \ \Delta_2 > 0, \ \Delta_3 > 0$$

$$\Rightarrow a_1 > 0, \ a_1 a_2 - a_3 > 0, \ a_3 \Delta_2 > 0$$

$$\Rightarrow a_1 > 0, \ a_3 < a_1 a_2, \ a_3 > 0.$$

$$\Rightarrow a_1 > 0, \ 0 < a_3 < a_1 a_2$$

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$$\Rightarrow a_1 > 0$$
$$a_3 > 0$$
$$a_2 > \frac{a_3}{a_1}$$

Remark: $a_1 > 0, a_2 > 0, a_3 > 0$ is a necessary but not sufficient condition for $P_3(\lambda)$ to have all zeros with negative real parts.

The Jury's stability criterion:

Let

$$P_n(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

be a polynomial with real co-efficients.

Theorem (Jury):—

Every zero of $P_n(\lambda)$ has magnitude less than one if and only if

Every zero of $(\lambda-1)^n \times P_n\left(\frac{\lambda+1}{\lambda-1}\right)$ has negative real parts.

Remark: Jury has reduced the problem to the Routh-Hurwitz problem.

Ex 3 $n=2$

$$P_2(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$$

$$P_2\left(\frac{\lambda+1}{\lambda-1}\right) =$$

$$a_0\left(\frac{\lambda+1}{\lambda-1}\right)^2 + a_1\left(\frac{\lambda+1}{\lambda-1}\right) + a_2$$

Define

$$Q_2(\lambda) = (\lambda-1)^2 P_2\left(\frac{\lambda+1}{\lambda-1}\right)$$

$$= a_0(\lambda+1)^2 + a_1(\lambda+1)(\lambda-1) + a_2(\lambda-1)^2$$

$$= (a_0 + a_1 + a_2) \times$$

$$\lambda^2 + \frac{2(a_0 - a_2)}{a_0 + a_1 + a_2}\lambda + \frac{a_0 - a_1 + a_2}{a_0 + a_1 + a_2}$$

Jury's Thm says that $P_2(\lambda)$ has zeros with magnitude < 1 iff $Q_2(\lambda)$ has zeros with negative real parts ie iff

$$\left.\begin{array}{l} \dfrac{2(a_0-a_2)}{a_0+a_1+a_2} > 0 \\[2ex] \dfrac{a_0-a_1+a_2}{a_0+a_1+a_2} > 0 \end{array}\right\} \text{Follows from example 1.}$$

$$\Rightarrow \left.\begin{array}{l} a_0 > a_2 \\ a_0+a_1+a_2 > 0 \\ a_0-a_1+a_2 > 0 \end{array}\right\} \text{OR} \left\{\begin{array}{l} a_0 < a_2 \\ a_0+a_1+a_2 < 0 \\ a_0-a_1+a_2 < 0 \end{array}\right.$$

$$\Rightarrow \left.\begin{array}{l} a_0 > a_2 \\ a_0+a_2 > |a_1| \end{array}\right\} \text{OR} \left\{\begin{array}{l} a_0 < a_2 \\ a_0+a_2 < -|a_1| \end{array}\right.$$

$$\Rightarrow$$

$$|a_1| - a_0 < a_2 < a_0$$

or

$$a_0 < a_2 < -|a_1| - a_0$$

If $a_0 = 1$, ie if $P_2(\lambda)$ is monic

then

$$a_0 < a_2 < -|a_1| - a_0$$

is never satisfied because

$$2a_0 < -|a_1|$$

is absurd. Hence we have

$$|a_1| - 1 < a_2 < 1$$

A necessary and sufficient condition for the polynomial

$$\lambda^2 + a_1\lambda + a_2$$

to have zeros with magnitude < 1 is that

$$|a_1| < a_2 + 1 < 2$$

I If $a_1 = \frac{1}{2}$ $a_2 = \frac{1}{2}$ we have

$$P_2(\lambda) = \lambda^2 + \frac{\lambda}{2} + \frac{1}{2} = 0$$

$$\Rightarrow \lambda = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2}}{2}$$

$$= -\frac{1}{4} \pm i\sqrt{\frac{7}{16}}$$

$$= \frac{-1 \pm i\sqrt{7}}{4} \qquad |\lambda| = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

Hence $|\lambda| < 1$

II If $a_1 = \frac{1}{2}$, $a_2 = 2$

$$P_2(\lambda) = \lambda^2 + \frac{\lambda}{2} + 2 = 0$$

$$\Rightarrow \lambda = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 8}}{2}$$

$$= -\frac{1}{4} \pm i\frac{\sqrt{31}}{4} = \frac{-1 \pm i\sqrt{31}}{4}$$

$$|\lambda| = \frac{\sqrt{32}}{4} = \frac{4\sqrt{2}}{4} = \sqrt{2} > 1.$$

III $a_2 = \frac{1}{4}$ $a_1 = \frac{3}{2}$. we have

$$P_2(\lambda) = \lambda^2 + \frac{3}{2}\lambda + \frac{1}{4} = 0$$

$$\Rightarrow \lambda = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - 1}}{2} = -\frac{3}{4} \pm \sqrt{\frac{5}{16}}$$

$$= \frac{-3 \pm \sqrt{5}}{4}$$

$\lambda = -.191, -1.31$

one root has magnitude > 1.